Journal of Applied Mechanics and Technical Physics, Vol. 47, No. 3, pp. 346–358, 2006

USING THE PERTURBATION METHOD TO SOLVE THE PROBLEM OF SEPARATED INCOMPRESSIBLE FLOW PAST THIN AIRFOILS

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UDC 532.5

A model for separated incompressible flow past thin airfoils in the neighborhood of the "shockless entrance" condition is constructed based on the averaging of the vortex shedding flow past the airfoil edges. By approximation of the vortex shedding by two vortex curves, determination of the average hydrodynamic parameters is reduced to a twofold solution of an integral singular equation equivalent to the equation describing steady-state nonseparated airfoil flow. In this case, the calculation time is two orders of magnitude smaller than the time required for the solution of the corresponding evolution problem. The results of a test calculation using the proposed method are in fair agreement with available results of calculations and experiments.

Key words: *incompressible fluid, airfoil, separation, vortex shedding.*

Introduction. Separated flows past bodies have been the subject of extensive research [1–3]. A classification of computational techniques for such flows is given in [2]. A separate group is distinguished that contains vortex models for inviscid flow using additional hypotheses such as the Joukowski—Chaplygin hypothesis, for which the main approaches to numerical implementation are described in [1]. The computational methods for separated flow past bodies considered in [1] are further developed in [3] taking into account fluid viscosity and using the boundary layer model.

The numerous examples of separated flow calculations given in [1, 3] and their comparison with experimental data show that the calculated velocity field is in good qualitative agreement with real fluid flows, and in a certain range of Reynolds numbers, good quantitative agreement between their hydrodynamic parameters is observed.

However, calculations of the formation and development of discontinuity surfaces modeling vortex shedding from airfoils in ideal fluid flow are laborious. Additional difficulties arise in flow averaging because separated airfoil flow is always unsteady and tends to steady-state flow only on the average [2].

The present paper deals with the development of a method for calculating separated incompressible flows past thin airfoils that is free from the above-mentioned drawbacks of existing computational methods for an asymptotic approximation of the solution of this problem.

1. Formulation of the Problem. We consider ideal incompressible flow over a curvilinear airfoil at an angle of attack

$$\alpha = \alpha_0 + \varepsilon, \qquad 0 < \varepsilon \ll 1, \tag{1.1}$$

where α_0 is the angle of "shockless entrance" at which the flow velocity in the neighborhood of the leading edge of the airfoil is limited.

Assuming that flow separation occurs only from the airfoil edges, we schematically represent the instantaneous position of the vortex shedding from the edges (Fig. 1), which can be calculated using the methods described in [1].

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 0021-8944/06/4703-0346

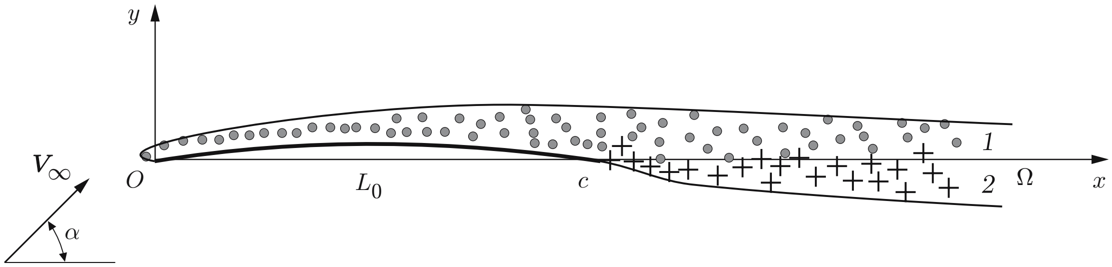

Fig. 1. Diagram of separated flow past a thin airfoil: 1 and 2 are free vortices shedding from the leading and rear edges of the airfoil.

In the ideal fluid model, the complex flow velocity in the plane $z = x + iy$ at each time t can be written as

$$\bar V(z,t) = \bar V_\infty + \frac{1}{2\pi i}\Big(\int\limits_{L_0} \frac{\gamma_0(\zeta_0,t)\,d\zeta_j}{z-\zeta_0(t)} + \sum_{j=1}^{2}\int\limits_{L'_j} \frac{\gamma_j(\zeta_j,t)\,d\zeta_j}{z-\zeta_j(t)}\Big), \qquad \bar V_\infty = q_\infty\,\mathrm{e}^{-i\alpha},$$

$$\zeta_j = \xi_j + i\eta_j, \qquad \zeta_0 \in L_0, \qquad \zeta_j(t) \in L'_j(t) \qquad \text{at} \quad j = 1,2,$$

where L_0 is the airfoil contour, L'_j $(j = 1,2)$ are the contours of the vortex shedding from the leading and rear edges, respectively, γ_j is the intensity per unit length of the vortex sheets modeling the airfoil contours L_0 and the vortex shedding L'_j.

The free vortex particles belonging to L'_j $(j = 1,2)$ pass through almost every point of a certain band Ω, whose boundaries are shown in Fig. 1 by solid curves. At the fixed points of the region Ω at the time t, the fluid vorticity $\omega(\zeta,t) \neq 0$ if $\zeta \in L'_j$ and $\omega(\zeta,t) = 0$ if $\zeta \notin L'_j$ $(j = 1,2)$. In view of this circumstance, the complex flow velocity at the time t can be written as

$$\bar V(z,t) = \bar V_\infty + \frac{1}{2\pi i}\Big(\int\limits_{L_0} \frac{\gamma_0(s_0,t)\,ds_0}{z-\zeta_0(s_0)} + \int\limits_{\Omega} \frac{\omega(\zeta,t)\,d\sigma}{z-\zeta}\Big), \tag{1.2}$$

where $\zeta = \xi + i\eta \in \Omega$, $d\sigma$ is an element of the region Ω, and s_0 is the arc coordinate L_0, with origin at the leading edge.

Assuming that in the neighborhood of the airfoil, the flow is on the average steady-state in time, we introduce the average quantities

$$\langle \bar V(z)\rangle = \frac{1}{T}\int\limits_0^T \bar V(t,z)\,dt, \quad \langle\omega(z)\rangle = \frac{1}{T}\int\limits_0^T \omega(t,z)\,dt, \quad \langle\gamma_0(s_0)\rangle = \frac{1}{T}\int\limits_0^T \gamma_0(t,s_0)\,dt \quad (T\to\infty). \tag{1.3}$$

According to (1.2), these quantities obey the relation

$$\langle \bar V(z)\rangle = \langle q(z)\rangle\,\mathrm{e}^{-i\langle\theta(z)\rangle} = \bar V_\infty + \frac{1}{2\pi i}\Big(\int\limits_{L_0} \frac{\langle\gamma_0(s_0)\rangle\,ds_0}{z-\zeta_0(s_0)} + \int\limits_{\Omega} \frac{\langle\omega(\zeta)\rangle\,d\sigma}{z-\zeta}\Big), \tag{1.4}$$

where Ω is a layer of finite thickness with the average particle vorticity.

The fluid flow considered satisfies the condition of nonpenetration through the contour L_0, which for the average flow can be written as

$$\mathrm{Im}\Big\{\langle \bar V(z_0)\rangle\,\mathrm{e}^{i\theta_0(z_0)}\Big\} = 0, \qquad z_0 \in L_0, \tag{1.5}$$

where $\theta_0(z_0)$ is the angle between the tangent to L_0 at the point z_0 and the Ox axis.

Substitution of (1.4) into (1.5) yields the integral equation

$$\int_{L_0} \langle\gamma_0(\zeta_0)\rangle K_0(z_0,\zeta_0)\,d\zeta_0 + \int_{\Omega} \langle\omega(\zeta)\rangle K_0(z_0,\zeta)\,d\sigma = 2\pi q_\infty \sin[\alpha - \theta_0(z_0)],$$

$$K_0(z_0,\zeta_0) = -\frac{(x_0-\xi_0)\cos\theta_0(z_0) + (y_0-\eta_0)\sin\theta_0(z_0)}{(x_0-\xi_0)^2 + (y_0-\eta_0)^2}, \qquad \zeta_0 \in L_0, \tag{1.6}$$

in which both the required function $\langle\gamma_0(\zeta_0)\rangle$ and the function $\langle\omega(\zeta)\rangle$ are unknown. The latter function is determined using the Joukowski–Chaplygin condition from formula (1.3) in solving the initially-boundary-value problem of separated airfoil flow.

Taking into account the assumption (1.1) that the deviation of the angle of attack α from the angle of "shockless entrance" α_0 is small, we solve the problem in question using a perturbation method [4]. For this, we write the required functions as the sum of two components

$$\langle\gamma_0(s_0)\rangle = \gamma_{00}(s_0) + \langle\gamma_0'(s_0)\rangle, \qquad \langle\bar V(z)\rangle = \bar V_0(z) + \langle\bar V'(z)\rangle, \tag{1.7}$$

where $\gamma_{00}(s_0)$ and $\bar V_0(z)$ are the required corresponding to the "shockless entrance" condition, $\langle\gamma_0'(s_0)\rangle$ and $\langle\bar V'(z)\rangle$ are their perturbed components for $\alpha = \alpha_0 + \varepsilon$. For the perturbed component of the fluid velocity, except in a small neighborhood of the leading edge of the airfoil, we assume

$$\langle q'(z)\rangle = O(\varepsilon), \qquad \langle\bar V'(z)\rangle = \langle q'(z)\rangle\, \mathrm{e}^{-i\langle\theta'(z)\rangle}. \tag{1.8}$$

The function $\gamma_{00}(s_0)$ satisfies the equation

$$\int_{L_0} \gamma_{00}(s_0) K_0(z_0,\zeta_0(s_0))\,ds_0 = 2\pi q_\infty \sin[\alpha_0 - \theta_0(z_0)], \qquad z_0 \in L_0, \tag{1.9}$$

whose solution is sought in the class of bounded functions on both ends of the contour L_0. For this class of solutions, the corresponding solvability condition for the equation can be treated as an equation for the angle of "shockless entrance" α_0.

According to perturbation method, the function $\langle\gamma_0'(s_0)\rangle$ should satisfy the equation

$$\int_{L_0} \langle\gamma_0'(s_0)\rangle K_0(z_0,\zeta_0)\,ds_0 + \int_{\Omega} \langle\omega(\zeta)\rangle K_0(z_0,\zeta)\,d\sigma = 2\pi q_{0\infty} \cos[\alpha_0 - \theta_0(z_0)]\varepsilon. \tag{1.10}$$

As shown below, the vortex sheet Ω in which the function $\langle\omega(\zeta)\rangle$ is defined can be modeled by two vortex curves, whose parameters depend only on two undetermined constants. In this model, the solution of Eq. (1.10) is found with accuracy up to the second order of smallness in ε.

2. Vortex Sheet Model Ω. Since the interactions of the vortex particles with each other and with the airfoil are different in nature, it is expedient to divide the vortex sheet Ω into three subregions Ω_j so that the coordinates x of the points belonging to it satisfy the conditions

$$\begin{aligned} x > c \qquad & \text{at } z \in \Omega_1, \\ 0 \leqslant x \leqslant c \qquad & \text{at } z \in \Omega_2, \\ -\tilde\varepsilon \leqslant x < 0 \qquad & \text{at } z \in \Omega_3. \end{aligned}$$

2.1. *Flow Model for the Subregion* Ω_1*.* In a linear approximation, it is usually assumed that the free vortex particles past a body in flow move at the main-flow velocity $\bar V_0(z)$. Figure 2 gives a schematic representation of a certain segment Ω_1. In the figure, the curves u and s show the natural curvilinear coordinates directed along main-flow streamlines and normal to them. The values of $u = 0$ and h define the upper and lower boundaries of the region Ω_1, the value of s_c is taken for $x = c$, and the hatched regions correspond to the vorticity diagram for $s = \text{const}$.

According to the Thompson theorem, the following equality holds:

$$\int_0^h \omega(s,u)\,du = 0. \tag{2.1}$$

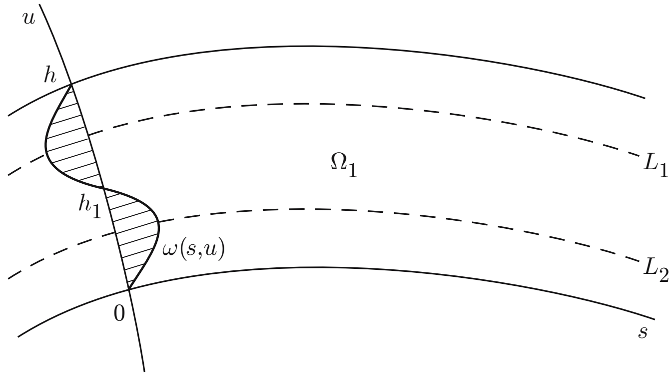

Fig. 2. Vortex sheet model past the airfoil.

Here and below, the angle brackets are omitted since they correspond only to the perturbed components of the flow parameters, which have their own notation. For simplicity without loss of generality, we assume that on the segment $(0, h)$, the value $\omega(s, u)$ vanishes not only at its ends but also at one point for a certain value of $u = h_1$. Then, according to (2.1) and in view of (1.8), we have

$$\int_0^{h_1} \omega(s,u)\,du = -\int_{h_1}^{h} \omega(s,u)\,du = \gamma_1(s) = O(\varepsilon q_\infty). \tag{2.2}$$

The integrals over the upper and lower parts of the region Ω_1 are considered equal to the average intensities per unit length of the vortices shedding from the leading and rear edges of the airfoil, respectively.

The multiple integral on the left side of (1.10) for $\zeta \in \Omega_1$ can be written in the form of the repeated integral

$$J_1 = \int_{s_c}^{\infty} \Big(\int_0^{h(s_c)} \omega(\zeta) K_0(z_0,\zeta)\,du \Big)\,ds.$$

The function $K_0(z_0, \zeta)$ is expended in a Taylor series in the variable u in the neighborhood of the value $u = 0$ for arbitrary $s > 0$:

$$K_0(z_0,\zeta) = K_0(z_0; s, 0) + \frac{\partial K_0}{\partial u}\Big|_{u=0} u + \dots\ .$$

Assuming that $h/c = O(\varepsilon)$ and taking into account (2.2), with accuracy up to terms of the second order of smallness we obtain

$$J_{11} = \int_0^h \omega(\zeta) K_0(z_0,\zeta)\,du = \frac{\partial K_0}{\partial u}\Big|_{u=0} \int_0^h \omega(s,u) u\,du.$$

For the centers of vorticity of the particles shedding from the leading and rear edges for fixed values of s_1, we write the expressions

$$u_{11}(s) = \int_{h_1}^{h} \omega(s,u) u\,du \Big/ \int_{h_1}^{h} \omega(s,u)\,du,$$

$$u_{12}(s) = \int_0^{h_1} \omega(s,u) u\,du \Big/ \int_0^{h_1} \omega(s,u)\,du,$$

and introduce the quantity that defines the distance between these centers:

$$\delta_1(s) = u_{11}(s) - u_{12}(s) = O(\varepsilon c). \tag{2.3}$$

Then, in view of (2.2), we obtain

$$J_{11} = -\gamma_1(s)\delta_1(s)\,\frac{\partial K_0}{\partial u}\Big|_{u=0}.$$

Thus, the vortex sheet Ω at $x > c$ is modeled by two vortex curves L_1 and L_2, which are discontinuity curves with values $-\gamma_1(s)$ and $\gamma_1(s)$, respectively, separated by a distance $\delta_1(s)$ from each other.

Using the relation

$$q_0(s)\gamma_1(s) = \text{const} \tag{2.4}$$

for the discontinuity curves and the relation

$$q_0(s)\delta_1(s) = \text{const},$$

which follows from the continuity equation, we obtain

$$\gamma_1(s) = \gamma_{1\infty} q_{0\infty}/q_0(s), \qquad \delta_1(s) = \delta_{1\infty} q_{0\infty}/q_0(s). \tag{2.5}$$

Here $q_0(s)$ is the modulus of the velocity of the main flow, whose complex velocity is defined by the formula

$$\bar{V}_0(z) = q_0(z)\,\mathrm{e}^{-i\theta(z)} = \bar{V}_{0\infty} + \frac{1}{2\pi i}\int\limits_{L_0} \frac{\gamma_{00}(s_0)\,ds_0}{z - \zeta_0(s_0)}, \qquad \bar{V}_{0\infty} = q_{0\infty}\,\mathrm{e}^{-i\alpha_0}, \tag{2.6}$$

$\gamma_{00}(s_0)$ is a solution of Eq. (1.9); $\gamma_{1\infty}$ is the intensity of the vortex sheet L_2 (the velocity discontinuity at L_2) and $\delta_{1\infty}$ is the distance between the curves L_1 and L_2 at infinity from the airfoil.

Therefore, the double integral on the left of Eq. (1.10) reduces the integral of a function of one variable dependent on two undetermined constants.

Let us show that using the model considered, we can estimate the loss of the total pressure due to the flow energy consumption in the vortex shedding process. For this, we use the Lamb–Gromeka equation for ideal fluid motion

$$\nabla\,(p + \rho q^2(z)/2) = \rho\,\boldsymbol{V} \times \boldsymbol{\omega}, \tag{2.7}$$

where the expression in brackets is the total pressure

$$P = p + \rho q^2(z)/2.$$

Because $\boldsymbol{\omega} = 0$ for $z \notin \Omega$, from Eq. (2.7) we have

$$P = P_{-\infty} = \text{const} \qquad \text{at} \quad z \notin \Omega. \tag{2.8}$$

We determine the projection of Eq. (2.7) onto the normal to the main-flow streamlines

$$\frac{\partial P}{\partial u} = \rho q(s,u)\omega.$$

Integrating this equation with allowance for (2.2) and (2.8), we obtain

$$P(z) = P_{-\infty} - \Delta P(z) \qquad \text{at} \quad z \in \Omega,$$

where $\Delta P(z)$ is the loss of the total pressure:

$$\Delta P(z) = \rho q_0(s)\gamma_1(s)[1 + O(\varepsilon)]. \tag{2.9}$$

It should be noted that using (2.5), from (2.9) we obtain the following expression for the airfoil drag due to the flow separation:

$$R_{x_1} = \rho q_{0\infty}\gamma_{1\infty}\delta_{1\infty}[1 + O(\varepsilon)], \tag{2.10}$$

which, according to [5], also follows from the momentum principle.

2.2. *Flow Model for the Subregion* Ω_2. Figure 3 schematically shows the vortex sheet shape in the neighborhood of the airfoil Ω_2, which corresponds to the visualized flow pattern for separated flow past thin airfoils [6] (the curve L_1 is the geometrical place of centers of vorticity of the average-flow particles, and its continuation is the corresponding curve in the region Ω_1; the curve L_3 is the upper boundary Ω_2). For convenience in describing the flow, we introduce a system of orthogonal curvilinear coordinates (s_0 and u_0), whose coordinate lines $u_0 = \text{const}$

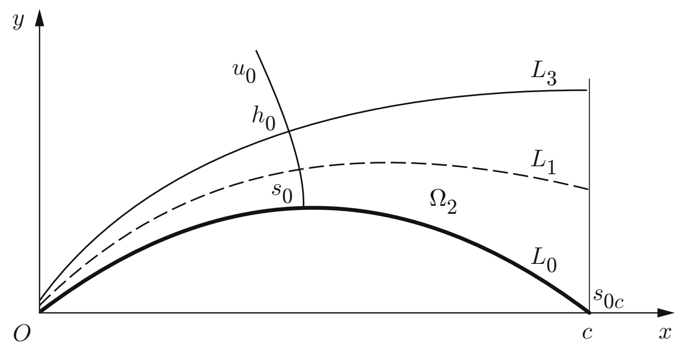

Fig. 3. Vortex sheet model over the airfoil.

coincide with the trajectories of the average motion of the vortex particles ($u_0 = 0$ on the airfoil contour L_0, and the value of s_0 coincides with the arc coordinate L_0). Similarly to (2.3), we write the multiple integrals on the left of Eq. (1.10) for Ω_2 as a repeated integral of the variables s_0 and u_0 and consider its internal integral over the variable u_0:

$$J_{21} = \int\limits_0^{h_0(s_0)} \omega(s_0, u_0) K_0(z_0, \zeta)\, du_0.$$

Here $h_0(s_0)$ is the coordinate u_0 of the upper boundary Ω_2.

With accuracy up to terms of the first order of smallness, the components of the complex coordinate $\zeta = \xi + i\eta \in \Omega_2$ can be written as

$$\xi = \xi_0 - u_0 \sin\theta_0(\zeta_0), \qquad \eta = \eta_0 + u_0 \cos\theta(\zeta_0), \qquad \zeta_0(s_0) \in L_0.$$

Substituting them into the expression for $K_0(z_0, \zeta)$, introduced in (1.6), we obtain

$$J_{21} = \int\limits_0^{h_0(s_0)} \omega(s_0, u_0)\, \frac{(x_0 - \xi_0)\cos\theta_0(z_0) + (y_0 - \eta_0)\sin\theta_0(z_0) + u_0 \sin[\theta_0(\zeta_0) - \theta_0(z_0)]}{(x_0 - \xi_0)^2 + (y_0 - \eta_0)^2 + 2u_0[(x_0 - \xi_0)\sin\theta_0(\zeta_0) - (y_0 - \eta_0)\cos\theta_0(\zeta_0)]}\, du_0.$$

Under the assumption that airfoil curvature $\varkappa(s_0) = O(1/c)$, this integral can be written as

$$J_{21}(s_0) = \int\limits_0^{h_0(s_0)} \omega(s_0, u_0)\Big(K_0(z_0, \zeta_0) + u_0[1 + O(\varepsilon)]\, \frac{1}{[(y_0 - \eta_0)^2 - (x_0 - \xi_0)^2]^2}$$

$$\times \Big\{[(y_0 - \eta_0)^2 - (x_0 - \xi_0)^2]\sin[\theta_0(\zeta_0) - \theta_0(z_0)]$$

$$+ 2(x_0 - \xi_0)(y_0 - \eta_0)\cos[\theta_0(\zeta_0) - \theta_0(z_0)]\Big\}\Big)\, du_0.$$

Similarly (2.2), we introduce a quantity $\gamma_1(s_0)$ that defines the average intensity per unit length of the vortices shedding from the leading edge and the coordinate of the centers of vorticity $u_0(s_0)$, which will be denoted by $\delta_2(s_0)$:

$$\gamma_1(s_0) = \int\limits_0^{h_0(s_0)} \omega(s_0, u_0)\, du_0, \qquad \delta_2(s_0) = \frac{1}{\gamma_1(s_0)} \int\limits_0^{h_0(s_0)} u_0 \omega(s_0, u_0)\, du_0.$$

Then, evaluating the integral $J_{21}(s_0)$ with accuracy up to terms of the second order of smallness, we obtain

$$J_{21}(s_0) = \gamma_1(s_0)\Big(K_0(z_0, \zeta_0) + \frac{\partial K_0(z_0, \zeta_0)}{\partial u_0}\, \delta_0(s_0)\Big).$$

Thus, like Ω_1, the vortex sheet Ω_2 can be modeled by the vortex curve L_1 to determine the average flow velocity with accuracy up to terms of the second order of smallness. However, the models of the vortex sheets Ω_1 and Ω_2 are different: according to Fig. 3, the vortex curve L_1 in the region Ω_2 cannot be located along the main-flow streamline since the source of the vortices is on the leading edge and the main-flow streamline passing through this edge coincides with L_0. The deviation of the trajectory of the vortex particles from the main-flow streamline can be due not only to the effect of the second approximation of the problem solution but also to the fluid viscosity and the unsteady nature of the particle separation process (especially in the neighborhood of the leading edge) since they occupy a minor part of the region Ω_2.

To determine the position of the curve L_1 with allowance for the flow continuity condition, we introduce the coordinates $u_0(s_0) = \delta_2(s_0)$ of its ends

$$\delta_2(0) = \varepsilon_0, \qquad \delta_2(s_{0c}) = \delta_1(0), \tag{2.11}$$

and the complex velocity of the average motion of the vortex particles

$$V_\omega(z) = q(z)\, \mathrm{e}^{i\theta'(z)}, \qquad z \in \Omega_2,$$

assuming that the modulus of this velocity is equal to the modulus of the average-flow velocity. The above-mentioned difference of the trajectory of the vortex particles from the average-flow streamlines will be taken into account using the function

$$\Delta\theta = \theta'(z) - \theta(z) = O(\varepsilon). \tag{2.12}$$

According to [7], the particle vorticity can be determined from the formula

$$\omega(z) = -\frac{\partial q(z)}{\partial u'} + \frac{q(z)}{R(z)}, \qquad \frac{1}{R(z)} = \frac{\partial \theta'(z)}{\partial s'}, \qquad z \in \Omega_2,$$

where $s'(z)$ is the direction of the average velocity of the vortex particles, $u'(z)$ is the direction of the normal to $s'(z)$, and $R(z)$ is the curvature radius of the vortex-particle trajectory.

At the same time, in the natural coordinate system, the vorticity equals

$$\omega(z) = -\frac{\partial q(z)}{\partial u} + q(z)\,\frac{\partial \theta(z)}{\partial s}$$

(u and s are the directions of the normal and tangent to the average-flow streamline). Eliminating $\omega(z)$ from these equations, converting to the total coordinate system (u, s) in the relation obtained, and using (2.12) and the continuity equation

$$q(z)\,\frac{\partial \theta(z)}{\partial u} + \frac{\partial q(z)}{\partial s} = 0, \tag{2.13}$$

we obtain

$$\frac{\partial}{\partial s}\,[\Delta\theta(z)] = O\Big(\frac{\varepsilon^2}{c}\Big).$$

The equations of the trajectories of average motion of the vortex particles can be written as relations between the coordinates $u_0'(z)$ and their arc coordinates s_0. The corresponding relations satisfy the equation

$$\frac{\partial u_0'(s_0)}{\partial s_0} = \theta'(z) - \theta_0(z_0) + O(\varepsilon^2).$$

Adding the function $\pm\theta(z)$ to the right side of this equation and taking the partial derivative with respect to s, we have

$$\frac{\partial}{\partial s}\Big(\frac{\partial u_0'(s_0)}{\partial s_0}\Big) = \frac{\partial}{\partial s}\,[\theta(z) - \theta_0(z_0)] + O\Big(\varepsilon^2\,\frac{q_\infty}{c}\Big).$$

Expanding the function $\theta(z)$ of this expression in a Taylor series in the neighborhood z_0 and using (2.13), we have

$$\frac{\partial^2}{\partial s_0^2}\,[u_0'(s_0)] = O\Big(\frac{\varepsilon^2}{c}\Big).$$

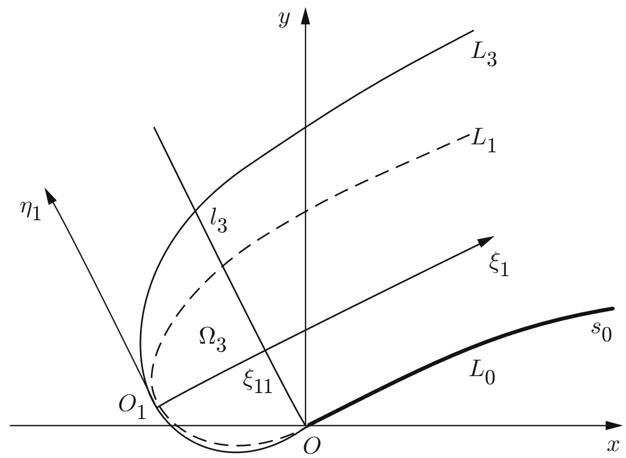

Fig. 4. Vortex sheet model ahead of the airfoil.

From this, for the required function $\delta_2(s_0)$, it is easy to derive the equation

$$\frac{\partial^2}{\partial s_0^2}\,[\delta_2(s_0)] = 0,$$

which defines the position of the curve L_1 in Ω_2 with accuracy up to terms of the first order of smallness. Solving this equation subject to conditions (2.11), we obtain

$$\delta_2(s_0) = C_0 s_0 + \varepsilon_0, \qquad C_0 = [\delta_1(0) - \varepsilon_0]/s_{0c}.$$

Below, it will be shown that for the quantity ε_0, the following estimate holds:

$$\varepsilon_0 = O(\varepsilon^2 c);$$

therefore, with accuracy up to terms of the first order of smallness and with allowance for (2.5), the position of the curve L_1 is defined by the formula

$$\delta_2(s_0) = \frac{\delta_{1\infty} q_{0\infty}}{s_{0c} q_0(s_{0c})}\, s_0. \tag{2.14}$$

2.3. *Flow Model for the Subregion* Ω_3. When the flow separates from the thin airfoil, the vortex particles shedding from the leading edge form a subregion of the vortex sheet Ω_3, which, according to the Joukowski–Chaplygin condition, is located in a small neighborhood ahead of this edge. By analogy with the region Ω_2, we model the vortex sheet in Ω_3 by a vortex curve L_1 (the dashed curve in Fig. 4), which at $s_0 = 0$ coincides with the corresponding curve in the region Ω_2. In this case, by virtue of the Joukowski–Chaplygin condition, the tangent to L_1 at the point $z = 0$ coincides with the tangent to L_0.

We introduce the arc coordinate s_1 of the curve L_1 with origin at the point of its conjugation with L_0 ($z = 0$) and a Cartesian coordinate system (ξ_1, η_1), whose $O_1\xi_1$ direction is parallel to the tangent to L_0 at the point $z = 0$, and whose $O_1\eta_1$ axis is tangent to L_1. The point of contact of the $O_1\eta_1$ axis and L_1 is the coordinate origin of this system.

Let us estimate the intensity per unit length $\gamma_1(s_1)$ of the vortex curves L_1 at the point $s_{10} = 0$ and the points s_{11} and s_{12} — the arc coordinates of the point of intersection L_1 with the axis $O\xi_1$ and the segment $l_3 \perp O\xi_1$ (boundary Ω_3), respectively. In view of (1.7), we note that the main-flow velocity corresponding to the velocity of the incident flow past the airfoil $\bar{V}_{0\infty}$ can be determined from the formula (2.6).

The perturbed component of the flow velocity arises from the additional flow incident on the airfoil at a velocity $\bar{V}'_\infty(z) = \varepsilon q_{0\infty} i\, e^{-i\alpha_0}$. The vortex curve L_1 can be treated as the free interface of a certain region Ω'_3 adjacent to the contour L_0 and to the fluid flow region external with respect to it.

Taking into account the Joukowski–Chaplygin condition, which can be interpreted as the nonpenetration condition for the initial segment L_1, by analogy with the jet models, we assume that

$$\bar{V}'(z) = 0, \qquad z \in \Omega'_3. \tag{2.15}$$

In this case, however, the curve L_1 is considered penetrable to the main flow, i.e., the Helmholtz equation describing free-vortex motion in an ideal fluid is not valid for the average flow in the region Ω_3. This circumstance can be treated as a consequence of the vortex formation due to flow separation from the leading edge (which leads to a loss of the total pressure); as a result, the condition of the Lagrange theorem are not satisfied in the region Ω_3 [5].

To justify the proposed flow model for the region Ω_3, we estimate some of its parameters. Postulating a loss of the total pressure in this region, which is a consequence of the Joukowski–Chaplygin condition and using (2.2), from (2.9) we obtain

$$\Delta P = O(\varepsilon \rho q_\infty^2). \tag{2.16}$$

In the model used, the total pressure changes suddenly by this value when intersecting the curve $L_1(s_1)$.

Let us consider the corresponding relation for the intersection of L_1 at the point $s_1 = s_{12}$:

$$p^+(s_{12}) + \rho(V_{s_1}^+)^2/2 - \Delta P = p^-(s_{12}) + \rho(V_{s_1}^-)^2/2.$$

Here p^+, $V_{s_1}^+$, p^-, and $V_{s_1}^-$ are the limiting values of the static pressure and the tangent component of the fluid velocity in approaching L_1 from above and from below, respectively. Taking into account the continuity condition of the static pressure on the free boundary and (2.16), from the above relation we obtain the estimate of the quantity γ_1 at the point $s_1 = s_{12}$ on the upper branch of L_1:

$$-\gamma_1(s_{12}) = V^+(s_{12}) - V^-(s_{12}) = O(\varepsilon q_{0\infty}). \tag{2.17}$$

Since $V^+(s_{12}) \approx q_0(s_{12}) - \gamma_1(s_{12})/2$, from (2.17) it follows that in the region Ω_3', the fluid velocity is on the order of the main-flow velocity, i.e., the curve L_1 is penetrable to the main flow and estimate (2.17) is consistent to (2.15).

Similarly, using the assumption (2.15), we obtain the following estimate of the quantity $\gamma_1(s_1)$ at the point $s_1 = s_{10} = 0$ on the lower branch L_1:

$$\gamma_1(s_{10}) = 2q_0(s_{10})[1 + O(\varepsilon)].$$

Below, the quantity of interest is the total intensity per unit length of the vortices of the lower and upper branches L_1 on the coordinate ξ at $\xi \to 0$.

According to [8], in a small neighborhood of the coordinate origin O_1 of the system (ξ_1, η_1), the intensity per unit length of the vortex curve L_1 is given by the expression

$$\gamma_1(\xi) = A/\sqrt{\xi}. \tag{2.18}$$

3. Algorithm for Determining Hydrodynamic Parameters. According to the Joukowski–Chaplygin condition, in the case of separated airfoil flow, $\gamma_0(s_0) < \infty$ for $s_0 \to 0$; therefore, the sucking force, which occurs for attached flow, tends to zero in the case considered. From this, it follows that the aerodynamic interaction of the airfoil with the flow separated from the airfoil edges is completely determined by the pressure gradient $\Delta p(s_0)$ on the airfoil contour. In particular, the projections of the resultant aerodynamic forces on the axes Ox_1 and Oy_1 can be defined by the formulas

$$R_{x1} = -\int\limits_{L_0} \Delta p(s_0) \sin\theta_1(s_0)\, ds_0, \qquad R_{y1} = \int\limits_{L_0} \Delta p(s_0) \cos\theta_1(s_0)\, ds_0,$$

where $\theta_1(s_0)$ is the angle between the tangent to L_0 at the point s_0 and the axis Ox_1. In the model considered above taking into account the continuity of the static pressure in passing through the vortex curve L_1, the following relation is valid:

$$\Delta p(s_0) = -\rho q(s_0)\gamma(s_0), \qquad s_0 \in L_0, \tag{3.1}$$

where

$$q(s_0) = q_\infty \cos[\alpha - \theta_0(s_0)] + \frac{1}{2\pi} \int\limits_{L_0} \gamma(\zeta_0) K_1(z_0, \zeta_0)\, d\zeta_0 + O(\varepsilon^2 q_\infty),$$

$$K_1(z_0, \zeta_0) = \frac{(x_0 - \xi_0)\sin\theta_0(z_0) - (y_0 - \eta_0)\cos\theta_0(z_0)}{(x_0 - \xi_0)^2 + (y_0 - \eta_0)^2}, \quad \gamma(s_0) = \gamma_{00}(s_0) + \gamma_0'(s_0) + \gamma_1(s_0).$$

It should be noted that (3.1) differs from the Joukowski formula in small in that apart from the functions of the attached-vortex intensity $\gamma_{00}(s_0)+\gamma_0'(s_0)$, the function $\gamma(s_0)$ contains the function of the free-vortex intensity $\gamma_1(s_0)$.

Let us show that with accuracy up to terms of the second order of smallness, the function $\gamma(s_0)$, which completely defines $\Delta p(s_0)$, can be found from Eq. (1.6), which, with the use of the model of the vortex sheet Ω considered in Sec. 2, is brought to the form

$$\int\limits_L \gamma(s_0)K_0(z_0,\zeta_0)\,ds_0 = F(z_0) + 2\pi q_\infty \sin[\alpha - \theta_0(z_0)],$$
$$z_0,\zeta_0 \in L = L_0 \cup \Delta L_0. \tag{3.2}$$

Here ΔL_0 is the segment of the tangent to L_0 at the point $z_0 = 0$ (of length l_0) between the point $z = 0$ and the point of its intersection with the axis $O_1\eta_1$;

$$F(z_0) = \int\limits_{L'} G(z_0,\zeta_0)\,ds_0, \qquad G = g(s_0)\frac{\partial K_0(z_0,\zeta)}{\partial u_0}\Big|_{\zeta=\zeta_0}, \qquad L' = L \cup L_2,$$

$$\begin{aligned} &\gamma(s_0) = \gamma_1^+ + \gamma_1^-, && g(s_0) = \gamma_1^+ u_0^+ + \gamma_1^- u_0^- && \text{at } -l_0 \leqslant s_0 < 0,\\ &\gamma(s_0) = \gamma_0' + \gamma_1 + \gamma_{00}, && g(s_0) = \gamma_1\delta_0 && \text{at } 0 \leqslant s_0 \leqslant s_{0c}; \end{aligned}$$

$\gamma_1^\pm$ and $u_0^\pm$ are the intensity and coordinates of points of the upper (superscript plus) and lower (superscript minus) branches of the vortex curve L_1, respectively; L_2 is the main-flow streamline segment with origin at the trailing edge of the airfoil.

It should be noted that Eq. (3.2) can be treated as the condition of fluid nonpenetration through the arc L. In this case, the fluid nonpenetration through the segment ΔL_0 — the frontal part of this arc — is a consequence of the Joukowski–Chaplygin condition for the smooth separation of the fluid from the leading edges of thin airfoils in separated flow.

We also note that apart from the sought-for function $\gamma(s_0)$, the function $F(z_0)$ on the right side of Eq. (3.2) is also unknown. We estimate the quantity $F(z_0)$. For this, taking into account (2.4) and (2.14), we write it as

$$F(\sigma) = q_{0\infty}^2\gamma_{1\infty}\delta_{1\infty}J(\sigma),$$

$$J(\sigma) = \int\limits_L \frac{f(s_0)}{q_0^2(s_0)}\,\frac{\partial K(z_0,\zeta_0)}{\partial u_0}\,ds_0, \qquad f(s_0) = \begin{cases} q_0(s_0)s_0/q(s_{0c})s_{0c}, & -l_0 \leqslant s_0 \leqslant s_{0c},\\ 1, & s_0 > s_{0c}. \end{cases}$$

For $z = \zeta_0$, the integrand $J(\sigma)$ has a singularity. To determine the order of magnitude of this singularity, we consider the function

$$\frac{\partial K_0(z_0,\zeta)}{\partial u_0}\Big|_{\zeta=\zeta_0} = \frac{1}{[(y_0-\eta_0)^2+(x_0-\xi_0)^2]^2}\Big\{[(y_0-\eta_0)^2+(x_0-\xi_0)^2]\sin[\theta_0(\zeta_0)-\theta_0(z_0)]$$
$$+\,2(x_0-\xi_0)(y_0-\eta_0)\cos[\theta_0(\zeta_0)+\theta_0(z_0)]\Big\}.$$

As $\zeta_0 \to z_0$ ($\zeta_0 \in L$, $z_0 \in L$), the following relations hold:

$$y_0 - \eta_0 = (x_0-\xi_0)\tan\theta_0(\zeta_0), \qquad \theta_0(z_0) = \theta_0(\zeta_0) + \frac{x_0-\xi_0}{R(z_0)\cos\theta_0(\zeta_0)},$$

where $R(z_0)$ is the curvature radius of the curve L at the point z_0. From this, it follows that

$$\frac{\partial K_0(z_0,\zeta)}{\partial u_0}\Big|_{\zeta=\zeta_0} = \frac{\cos\theta_0(z_0)}{R(z_0)(x_0-\xi_0)},$$

and the integral $F(z_0)$ is a Cauchy integral, which is evaluated in the sense of the principal value. According to (2.18), at the left end of the integration contour (as $s_0 \to -l_0$), the density of this integral has a power-law singularity of order 1/2. Therefore, taking into account that for $s_0 \geqslant c$, we have the estimate

$$\frac{\partial K_0(z_0,\zeta_0)}{\partial u_0}\Big|_{\zeta=\zeta_0} = O\Big(\frac{c^2}{s_0^2}\Big),$$

and following [9], we find that the integral $J(\sigma)$ is a bounded function everywhere and the quantity $F(\sigma)$, in view of (1.8) and (2.3), has the estimate

$$F(\sigma) = O(\varepsilon^2 q_\infty). \tag{3.3}$$

In view this circumstance, the function $\gamma(s_0)$ is written as the sum of the two components

$$\gamma(s_0) = \gamma_{11}(s_0) + \gamma_{12}(s_0), \tag{3.4}$$

which should satisfy the equations

$$\int\limits_L \gamma_{11}(s_0) K_0(z_0, \zeta_0)\, ds_0 = 2\pi q_\infty \sin[\alpha - \theta_0(z_0)]; \tag{3.5}$$

$$\int\limits_L \gamma_{12}(s_0) K_0(z_0, \zeta_0)\, ds_0 = F(z_0). \tag{3.6}$$

From the expression for the kernel $K_0(z_0, \zeta_0)$ of these equations obtained in the derivation of Eqs. (1.6) and (1.9), it follows that Eqs. (3.5) and (3.6) are singular first-order integral equations which are equivalent to the equation describing steady-state nonseparated flow past curvilinear thin airfoils. According to (2.18), the solution of these equations is sought in the class of functions that are not bounded at the end $s_0 = -l_0$ of the contour $L = L_0 \cup \Delta L_0$ and are bounded at the end $s_0 = s_{0c}$. In this case, we first seek a solution of Eq. (3.5), whose right side is determined, and then a solution of Eq. (3.6), whose right side, as will be shown below, is determined with accuracy up to terms of the second order of smallness using the solution of Eq. (3.5).

We consider the solution of Eq. (3.5). Its form allows the function $\gamma_{11}(s_0)$ to be treated as the intensity of the vortex sheet modeling the contour L in nonseparated flow incident at an angle of attack α.

According to the hydrodynamic thin-airfoil theory [8], the quantities

$$(R_L)_{x_1} = \rho \int\limits_L q_{11}(s_0)\gamma_{11}(s_0) \sin\theta_1(s_0)\, ds_0, \qquad (R_L)_{y_1} = -\rho \int\limits_L q_{11}(s_0)\gamma_{11}(s_0) \cos\theta_1(s_0)\, ds_0$$

are components of the vector $\boldsymbol{R}_L$:

$$\boldsymbol{R}_L = \boldsymbol{R}_J - \boldsymbol{Q}$$

($\boldsymbol{R}_J$ is the Joukowski force vector and $\boldsymbol{Q}$ is the sucking force vector). The projections of these vectors onto the axes Ox_1 and Oy_1 are equal to

$$(R_J)_{x_1} = 0, \qquad (R_J)_{y_1} = -\rho q_\infty \Gamma_{11}, \qquad \Gamma_{11} = \int\limits_L \gamma_{11}(s_0)\, ds_0,$$

$$(R_Q)_{x_1} = -(\pi/4)\rho A^2 \cos[\alpha - \theta_0(0)], \qquad (R_Q)_{y_1} = (\pi/4)\rho A^2 \sin[\alpha - \theta_0(0)].$$

Here A is the coefficient in expression (2.18) for the singularity of the function $\gamma_{11}(s_0)$ at the leading edge L, which in the notation of Eq. (3.2) is written as

$$\gamma_{11}(s_0) = A/\sqrt{s_0 + l_0}, \qquad s_0 \in L. \tag{3.7}$$

This implies that

$$(R_L)_{x_1} = (\pi/4)\rho A^2 \cos[\alpha - \theta_0(0)], \quad (R_L)_{y_1} = -\rho\{q_\infty \Gamma_{11} + (\pi/4)A^2 \sin[\alpha - \theta_0(0)]\}. \tag{3.8}$$

By virtue of a small deviation of the angle α from the angle of “shockless entrance” α_0 ($\alpha - \alpha_0 = \varepsilon$), the coefficient A in expression (3.7) has the estimate

$$A/(q_\infty \sqrt{c}) = O(\varepsilon), \tag{3.9}$$

which, in view of (2.10) and (3.8), agrees with (3.3). By analogy with (3.4), the components of the sought-for force vector is written as

$$R_{x_1} = R_{x_1,1} + R_{x_1,2}, \qquad R_{y_1} = R_{y_1,1} + R_{y_1,2}. \tag{3.10}$$

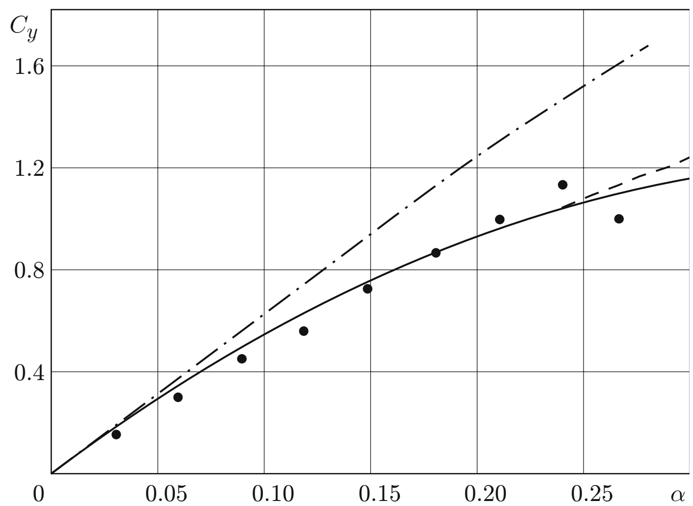

Fig. 5. Results of test calculation: the solid curve is calculated using the proposed method; the dash-and-dotted curve is calculated using attached-flow theory; the dashed curve is the average value of C_y obtained by solving the evolution problem [1]; the points are the results of experiments for an airfoil with the geometrical characteristics similar to the geometrical characteristics of a plate [3].

Using (3.9), from (3.8), we obtain

$$R_{x_1,1} = 0, \qquad R_{x_1,2} = (\pi/4)\rho A^2 \cos[\alpha - \theta_0(0)][1 + O(\varepsilon)]. \tag{3.11}$$

It should be noted that the quantity $(R_L)_{y_1}$ considered above differs from the corresponding component of the force R_{y_1} acting on the airfoil contour L_0 since the contour L differs somewhat from the airfoil L_0. Taking into account that $L = L_0 \cup \Delta L_0$, we consider the part of the quantity $(R_L)_{y_1}$ that corresponds to the segment ΔL_0 of the contour L:

$$\Delta(R_L)_{y_1} = -\rho q_0(0)\Delta\Gamma, \qquad \Delta\Gamma = \int\limits_{\Delta L_0} \gamma_{11}(s_0)\, ds_0.$$

Using (3.6), we obtain $\Delta\Gamma = 2A\sqrt{l_0}$. Because on the segment ΔL_0, the function $\gamma(s_0) = \gamma_1^+(s_0) + \gamma_1^-(s_0)$, from (2.17) and (2.18), we obtain

$$\sqrt{l_0} = \frac{A}{2q_0(0)}\,[1 + O(\varepsilon)].$$

As a result, we have

$$\Delta(R_L)_{y_1} = -\rho A^2[1 + O(\varepsilon)]. \tag{3.12}$$

[In view of (3.9), the expression for $\sqrt{l_0}$ leads to the estimate $\delta_0(0) = \varepsilon_0 = O(\varepsilon^2 c)$, which *a priori* was used in the derivation of formula (2.14).]

Let us consider Eq. (3.6). With allowance for (2.10) and (3.11), the right side of this equation is written with accuracy up to terms of the second order of smallness in the form

$$F(z_0) = -(\pi/4)A^2 \cos[\alpha - \theta_0(0)]J(z_0).$$

Using the solution of this equation from (3.1), we find the lifting force component

$$R'_{y_1,2} = -\rho \int\limits_{L_0} [q_{11}(s_0)\gamma_{12}(s_0) + q_{12}(s_0)\gamma_{11}(s_0)] \cos\theta_1(s_0)\, ds_0. \tag{3.13}$$

As a result, from (3.8)–(3.13) we obtain the following relations with accuracy up to terms of the second order of smallness:

$$R_{x_1} = (\pi/4)\rho A^2 \cos[\alpha - \theta_0(0)] = |Q| \cos[\alpha - \theta_0(0)];$$

$$R_{y_1} = -\rho\{q_\infty \Gamma_{11} + R'_{y_1,2} - A^2[(\pi/4)\sin[\alpha - \theta_0(0)] + 1]\}. \tag{3.14}$$

Thus, in the model considered, the time of calculation of the average aerodynamic characteristics of thin airfoils is two orders of magnitude smaller than that in the methods based on the solution of the evolution problem [1]. It should be noted that the aerodynamic drag of thin airfoils due to flow separation from their edges is proportional to the sucking force, which is present in the model of attached flow past the airfoil.

4. Verification of the Proposed Model. The geometrical model of the vortex shedding and the estimate of its parameters given in Sec. 2 are in good qualitative agreement with the observation results presented in [6]. A comparison of the quantitative results of calculation of the aerodynamic characteristics of thin airfoils for the given model with the results of calculation using another method and with experimental data was performed for the case of flow past a plate. Figure 5 gives a curve of the lift coefficient of the plate C_y versus the angle of attack α. It should be noted that in the case considered, where $\alpha_0 = 0$, $\theta_0(s_0) = 0$, and $\alpha = \varepsilon$, the curve of $C_y(\alpha)$ plotted by formula (3.14) invoking thin-airfoil theory [8] is transformed to an analytical expression of the form

$$C_y = 2\pi \sin\varepsilon(1 - (4/\pi)\sin\varepsilon).$$

From Fig. 5, it follows that the calculation results for the given model agree satisfactorily with experimental data over a wide range of the angle of attack.

This work was supported by the Integration project of the Siberian Division of the Russian Academy of Sciences (Project No. 27).

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